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The effect of mean shear flows on zonal flow formation is considered in the contexts of plasma drift wave turbulence and quasi-geostrophic turbulence models. The generation of zonal flows by modulational instability in the presence of large-scale mean shear flows is studied using the method of characteristics as applied to the wave kinetic equation. It is shown that mean shear flows reduce the modulational instability growth rate by shortening the coherency time of the wave spectrum with the zonal shear. The scalings of zonal flow growth rate and turbulent vorticity flux with mean shear are determined in the strong shear limit. © *2015 AIP Publishing LLC.*
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I. INTRODUCTION

Shearing structures are commonly observed in laboratory plasmas, planetary atmospheres, and the solar interior. In many of the systems two kinds of shear flows co-exist: (1) zonal flows which are generated by and coupled to turbulence[1,2] and (2) mean shear flows which are driven by background gradients or external stresses.[3] Turbulence, zonal flows and mean shears, all have effects on one another. The role of shearing in de-correlating turbulent eddies and regulating turbulent transport[1–5] is well known. Besides the shearing effects on turbulence, mean shear flows also affect the formation of zonal flows. In this paper, we study the effect of mean shear flows on modulational instability generation of zonal flows. In particular, we use wave kinetics to investigate quantitatively how a mean shear affects the correlation between turbulent wave packets and the zonal flow shearing field during modulational instability. Zonal flows are generated by modulational instability, in which the modulations due to a seed zonal flow induce variation in the fluctuation wave spectrum, and the Reynolds stress driven by this modulated (i.e., tilted) wave spectrum amplifies the initial perturbation. By de-correlating the modulated wave packet and the zonal shear, mean shears inhibit the growth of zonal flows.

Magnetically confined plasma is one of the systems in which mean shear flows, zonal flows, and drift wave turbulence co-exist and interact with one another. In tokamaks, the reduction of turbulence by flow shearing is believed to be a key mechanism for the low to high (L–H) plasma confinement mode transition, in which plasmas organize themselves into a high confinement state (H mode) by the formation of transport barriers. Of critical importance to the research of the L–H transition is the understanding of the interplay among mean and zonal shearing fields and drift wave packets. Zonal flows and mean shears play different roles in the L–H transition, because they differ in their generation mechanism, temporal behavior, and spatial structure. Zonal flows are generated by turbulence via the Reynolds stress while mean $E \times B$ flows are driven by the pressure gradient. Therefore, zonal flows must eventually decay when the underlying turbulence drive is extinguished, while mean shear flows can be sustained in the absence of turbulence. The shearing of zonal flows has a complex spatial structure and is of limited coherency in time while the shearing by mean flows is coherent over longer times. With respect to their spatial scales, the scale of mean shears is macroscopic, comparable to the characteristic scale lengths of the system profiles, while the scale of zonal flows is mesoscopic, between the micro-scale turbulence correlation length and macro-scale system size. The zonal flow and drift wave packet both have a mesoscale character: L_P^{-1} < meso wavenumber < k, where L_P is the profile scale and k is the turbulence wave number. An illustration of the multi-scale system with macro-scale mean shears, meso-scale zonal flows and drift wave packets is shown in Fig. 1. In the predator-prey model for the L–H transition,[6,7] the transition is triggered by zonal flows regulating turbulence and lowering the power threshold. By extracting kinetic energy from drift wave turbulence, zonal flows regulate the turbulence level and associated transport, allowing the buildup of a steep pressure gradient. During the transition, the self-regulation of turbulence by zonal flows causes an oscillatory temporal behavior. Then, as the mean shear grows sufficiently strong, both turbulence and zonal flows are damped at the final stage
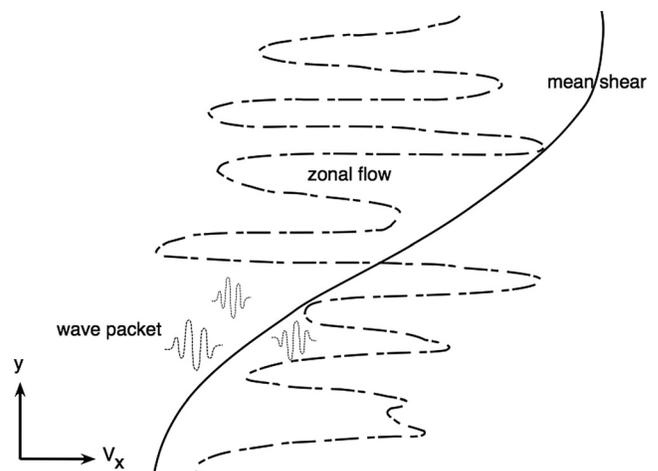
FIG. 1. Multi-scale system.

of the L–H transition. Thus, the role of mean shear flows in the L–H transition is not only to de-correlate turbulence but also to regulate zonal flows. The comprehension of the L–H transition requires an understanding of the coupled dynamics of a system of turbulence, zonal flows, and mean shears. In this paper, we present a quantitative study of one of the key issues: the effect of a mean shear flow on the modulational instability of zonal flows.

Another system where mean shears may have significant effect on turbulent momentum transport associated with zonal flows is the solar tachocline. Helioseismology reveals that the angular velocity varies with latitude in the convection zone while the rotation in the radiation zone is nearly uniform. Between the convective envelope and radiative interior there exists a thin transition layer (less than 5% of the solar radius) known as the tachocline. It is still unclear how the tachocline remains thin under the force of the overlying differential rotation of the convection zone. Spiegel and Zahn[8] attribute the thin tachocline to anisotropic turbulence. They argue that turbulence in the stably stratified tachocline has negligible vertical motion and hence acts like a large constant horizontal viscosity producing mixing on spherical surfaces. The anisotropic turbulent viscosity in their model diffuses the latitudinal differential rotation, prevents the spreading of differential rotation into the radiative interior, and keeps the tachocline thin. However, the effect of the large scale differential rotation on turbulent momentum transport is not considered in the Spiegel-Zahn model. Moreover, turbulent momentum transport coefficients should probably depend on shearing flows, since turbulence in the tachocline is coupled to both turbulence-generated flows and large scale rotation shears. Therefore, models for momentum transport in the tachocline should consider the interplay between solar differential rotation, zonal flows, and turbulence and waves. Similar to the mean $E \times B$ shears in tokamaks, differential rotation in the solar tachocline can be treated as a stable, imposed macro-scale mean shear because it is maintained by large-scale stresses from the convective zone above and its evolution timescale is much longer than that of turbulence and turbulence-driven zonal flows. We study the influence of the meridional differential rotation on turbulent momentum transport in a 2D quasi-geostrophic fluid. The effect of the radial differential rotation on turbulent transport in the solar tachocline is investigated by Kim.[9] Note that the model in this paper is purely hydrodynamic. Magnetic fields may exist and have important effects on the formation of the tachocline and transport in the tachocline. There also are MHD models of the tachocline. Gough and McIntyre[10] argue that a fossil magnetic field in the radiative interior is required to prevent the spreading of the shear, since turbulence in the tachocline would mix potential vorticity (PV) and drive shear flows. Therefore, the key problem remains to be how turbulence in the tachocline acts to redistribute angular momentum, i.e., what the form of turbulent momentum (or PV) flux is in the tachocline. In the β-plane MHD model by Tobias et al.,[11] magnetic field is found to suppress turbulent momentum transport and the generation of mean flows in the tachocline. An MHD model of the

mean shear-zonal flow-turbulence coupling system will be considered in future work.

In this paper, we examine the effect of mean shear flows on the modulational instability and growth of zonal flows. We consider systems in which macro-scale mean shear and meso-scale zonal shear coexist and discuss the effects of fixed zonal mean shears on turbulent momentum transport and the coupling between different scale shearing fields. We show that mean shear flows reduce the turbulent momentum transport which forms zonal flows by decorrelating wave packets and the zonal flow shearing field during the process of modulation. The de-correlation by mean shear during the modulation also reduces the growth rate of the zonal flow. The scalings of turbulent momentum flux and zonal flow growth rate with a strong mean shear Ω are both $\Omega^{-2/3}$.

Previous work has explored the effect of mean shears on reducing the growth of the modulational instability (see, e.g., Refs. [12] and [13]), following the initial work by Kim and Diamond.[6] However, this paper makes a significant advance. Specifically, this paper includes (1) an analysis of the effect of a mean shear on modulational wave action density $\tilde{N}_k$, zonal flow growth rate γ_q, and turbulent viscosity ν_t; (2) a study of zonal/mean flow shearing, PV mixing and their interaction in modulational instability; (3) a calculation of the spatial flux of PV. Of course, a lot more needs to be done to completely unravel the physics of drift/Rossby wave-zonal flow turbulence in tokamaks and the tachocline.

The organization of the remainder of this paper is as follows. Section II gives an introduction to zonal flow formation via modulational instability without the presence of mean shearing field. The effect of ambient mean shear flows on zonal flow generation is studied in Sec. III. Section IV presents our conclusion and discussions.

II. ZONAL FLOW GENERATION VIA MODULATIONAL INSTABILITY

Before considering the effect of mean shears on the generation of zonal flows, we first briefly review the modulational instability of the zonal flow formation.[1,14] The systems we are interested in are drift wave turbulence in magnetically confined plasmas and quasi-geostrophic turbulence in geophysical fluid dynamics (GFD). Both systems are approximately two dimensional because of the strong guiding field applied to magnetized plasmas and the rapid planetary rotation and strong density stratification in the quasi-geostrophic limit. The model equations of these two systems are the Hasegawa-Mima (HM) equation[15] and the quasi-geostrophic equation.[16] The Hasegawa-Mima equation for drift wave turbulence is given by

$$\frac{1}{\omega_{ci}} \frac{\partial}{\partial t} \left(\nabla^2 \psi - \rho_s^{-2} \phi \right) - \frac{1}{L_n} \frac{\partial}{\partial y} \phi + \frac{1}{\rho_s L_n} J\left(\phi, \nabla^2 \phi\right) = 0, \quad (1)$$

where ω_{ci} is the ion cyclotron frequency, ϕ is the normalized electrostatic potential, ρ_s is the ion gyroradius at electron temperature, L_n is the density gradient scale length, y-axis is in the poloidal direction, and J is the Jacobian operator. The 2D quasi-geostrophic equation is given by

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - L^{-2}\psi\right) + \beta\frac{\partial}{\partial x}\psi + J\left(\psi, \nabla^2\psi\right) = 0, \quad (2)$$

where ψ is the stream function, L is the Rossby deformation radius, β represents the latitudinal variation of the Coriolis parameter, and x-axis is in the zonal direction. The Hasegawa-Mima equation and the quasi-geostrophic equation have the same structure. Both equations express material conservation of PV in the inviscid limit. We use the GFD notation and Eq. (2) for the rest of this paper, so the y-axis is in the direction of inhomogeneity: the radial direction in plasma or the meridional direction in GFD, and the x-axis is in the direction of symmetry, i.e., the direction of the zonal flows. Here, we consider the turbulence scales which are much smaller than L or ρ_s. In this limit, PV is $\nabla^2\psi + \beta y$ ($\beta = \frac{\partial}{\partial y}\ln n_0$ for drift wave turbulence). The flux of PV is simply the flux of vorticity, and the dispersion relation of the linear waves (drift waves in plasma or Rossby waves in GFD) is $\omega_k = -\beta k_x/k^2$, where $k^2 = k_x^2 + k_y^2$. Note that planetary vorticity βy varies in $\hat{y}$. As a consequence, the relative vorticity $\nabla^2\psi$ of a fluid parcel must change accordingly as it moves along the meridional direction in order to conserve PV, resulting in the propagation of a Rossby wave.

The physics of zonal flow generation by drift wave modulational instability is contained in the Hasegawa-Mima equation. However, it is recognized that there are many more complex dynamical models in magnetically confined fusion plasmas (e.g., ion temperature gradient (ITG), electron temperature gradient (ETG), trapped electron mode (TEM)).[17–19] We note that all of ITGs, TEMs, ETGs, etc., are drift waves, in which ion (or electron, for ETG) polarization and finite-Larmor-radius (FLR) effects act as inertia which increases the dielectric, i.e., $\epsilon = 1 + k_\perp^2\rho_s^2 + ...$, which then modifies (lowers) the wave frequency. In all cases the frequency is in the range of the drift wave frequency, and therefore can be written as $\omega \simeq \omega_{*e,i}f(k_\perp\rho_s)$ (or $f(k_\perp\rho_e)$, for ETG), where function $f(k_\perp\rho_s) < 1$. As $k_\perp\rho_s \to 0$, $\epsilon = 1 + k_\perp^2\rho_s^2 + ... \to 1 + ...$, and so $f(k_\perp\rho_s) \to 1$. For example, the generic drift wave frequency is given as $\omega = \omega_{*i}/(1 + k_\perp^2\rho_s^2)$. For the collisionless trapped electron mode (CTEM), the frequency is reduced as $\omega = \omega_{*e}/(1 + k_\perp^2 k_M^{-2})$, where $k_M^{-2} = \rho_s^2[1 + \tau^{-1}(1 + \eta_i)]$, $\tau = \frac{T_e}{T_i}$, and $\eta_i = \frac{\partial_r\ln T_i}{\partial_r\ln n_i}$. In the case of the ITG mode, the influence of the ion temperature gradient is coupled with parallel ion motion, and the dispersion relation, taking account of the ion Larmor radius, is given as $\omega^2 - \omega\omega_{*e}/(1 + k_\perp^2\rho_s^2(1 - \tau^{-1})) - k_\parallel^2 c_s^2(1 + \tau^{-1}\eta_i\frac{\omega_{*e}}{\omega}) = 0$. The ETG mode is similar to the ITG mode, but with the roles of the electrons and ions reversed. In the ITG and TEM modes, we have the Boltzmann electron response for both waves and zonal mode, while the ion response to zonal flow perturbations is adiabatic in the ETG mode. Thus, the simplified model discussed in our paper captures the basic trends of the dependence of the dielectric and wave frequency on the polarization and FLR effects, which are common to all models, however, more complicated.

In all cases one can understand the energy transfer between drift waves and zonal flows by exploiting the mean field evolution equation of the wave action density[1]

$$\frac{d\langle N\rangle}{dt} = \frac{\partial}{\partial k_r}D_{k_r}\frac{\partial\langle N\rangle}{\partial k_r}. \quad (3)$$

Multiplying Eq. (1) by ω_k and proceeding to integrate by parts gives the evolution of the wave energy

$$\frac{d\varepsilon_{wave}}{dt} = -\int D_{k_r}\frac{\partial\omega_k}{\partial k_r}\frac{\partial\langle N\rangle}{\partial k_r} = -\int D_{k_r}v_{gr}\frac{\partial\langle N\rangle}{\partial k_r}, \quad (4)$$

where the group velocity $v_{gr} = \partial\omega_k/\partial k_r$ can be derived from the dielectric function $\epsilon(\omega, \boldsymbol{k})$, since $d\epsilon = \frac{\partial\epsilon}{\partial\omega}d\omega + \frac{\partial\epsilon}{\partial\boldsymbol{k}}\cdot d\boldsymbol{k} = 0$ along the wave path. Therefore, the wave energy evolves (in terms of ϵ)

$$\frac{d\varepsilon_{wave}}{dt} = \int D_{k_r}\frac{\partial\epsilon/\partial k_r}{\partial\epsilon/\partial\omega}\bigg|_{\boldsymbol{k},\omega_{\boldsymbol{k}}}\frac{\partial\langle N\rangle}{\partial k_r}. \quad (5)$$

For drift wave turbulence, $d\varepsilon_{wave}/dt < 0$ (i.e., the wave energy decreases and so the flow energy grows) since the conditions $k_r\partial\langle N\rangle/\partial k_r < 0$ and $v_{gr}/k_r < 0$ are virtually always satisfied. The condition $v_{gr}/k_r < 0$ results from the generic structure of ϵ: that the sign $k_r^{-1}(\partial\epsilon/\partial k_r)$ and $\partial\epsilon/\partial\omega$ are the same generically (e.g., $\epsilon(\omega, k_\perp\rho_s) = 1 - \frac{\omega_{*i}}{\omega(1 + k_\perp^2\rho_s^2)}$). Thus, the growth of zonal flows at the expense of the wave energy is generic to all of the drift wave models and follows from the basic structure of ϵ. The simple model presented in this paper captures the basic physics of wave-flow energy transfer of the drift wave-zonal flow systems. While the details of $\epsilon(\omega, k_\perp\rho_s)$ or $f(k_\perp\rho_s)$ may change, or be more complicated, the trend that turbulence loses energy to the flows persists.

The modulational instability of a broad spectrum of Rossby or drift waves to a large-scale test zonal flow δV_x is given by

$$\frac{\partial}{\partial t}\delta V_x = -\frac{\partial}{\partial y}\langle\tilde{v}_y\tilde{v}_x\rangle - \mu\delta V_x$$
$$= \frac{\partial}{\partial y}\int d^2 k k_x k_y|\tilde{\psi}_k(\delta V_x)|^2 - \mu\delta V_x, \quad (6)$$

where μ is friction in GFD or collisional drag in plasma. Note that the Taylor identity relates the Reynolds force to the cross zonal stream flux of vorticity, indicating vorticity (or PV) transport and mixing in a system with one direction of symmetry as the fundamental mechanism of zonal flow formation in a quasi-2D fluid or plasma. We use wave action (population) density conservation to evaluate the modulational response of the wave spectrum to the test shear. The wave action density is given by $N_k = \varepsilon_k/\omega_k$, where ε_k is the wave energy density. In the case of drift wave turbulence, the wave energy density is given by $\varepsilon_k = (1 + k_\perp^2\rho_s^2)|\psi_k|^2$ and so $N = (1 + k_\perp^2\rho_s^2)^2|\psi_k|^2/\omega_*$. For zonal flow shears, ω_* is unchanged by flow shearing, so the wave action density and the potential enstrophy density $(1 + k_\perp^2\rho_s^2)^2|\psi_k|^2$ are identical, up to a constant factor. In the systems considered in this work, we normalize the wave action density $N_k = -k^4|\psi_k|^2/(\beta k_x)$ to the enstrophy density $k^4|\psi_k|^2$, since β is a constant and k_x is unchanged by zonal flow shearing. Thus, we see that the wave action density represents the intensity

field of PV and its evolution has a direct connection with PV mixing. The evolution of zonal flow is determined by the modulational response of enstrophy density $\tilde{N}_k$, as

$$\frac{\partial}{\partial t}\delta V_x = \frac{\partial}{\partial y}\int d^2k\frac{k_xk_y}{k^4}\tilde{N}_k - \mu\delta V_x$$

$$= \frac{\partial}{\partial y}\int d^2k\frac{k_xk_y}{k^4}\left(\frac{\partial\tilde{N}_k}{\partial\delta V_x}\right)\delta V_x - \mu\delta V_x. \qquad (7)$$

N_k is determined by the linearized wave kinetic equation (WKE)

$$\frac{\partial\tilde{N}_k}{\partial t} + v_{gy}\frac{\partial}{\partial y}\tilde{N}_k + \delta\omega_k\tilde{N}_k = \frac{\partial(k_x\delta V_x)}{\partial y}\frac{\partial N_0}{\partial k_y}, \qquad (8)$$

where N_0 is the mean enstrophy density. $\delta\omega_k$ represents nonlinear self-decorrelation rate via wave-wave interaction. The equilibrium balance has been used to relate linear growth rate and nonlinear damping rate. For small perturbations $(\tilde{N}_k, \delta V_x) \sim e^{-i\Omega_q t + iqy}$, the modulation of $\tilde{N}_k$ becomes

$$\tilde{N}_k = iq\delta V_x\frac{k_x}{-i(\Omega_q - qv_{gy}) + \delta\omega_k}\frac{\partial N_0}{\partial k_y}, \qquad (9)$$

so the growth rate of the test zonal flow is given by

$$\gamma_q = -q^2\int d^2k\frac{k_x^2k_y}{k^4}\frac{|\delta\omega_k|}{(\Omega_q - qv_{gy})^2 + \delta\omega_k^2}\frac{\partial N_0}{\partial k_y}. \qquad (10)$$

The condition to have instability $(k_y\partial N_0/\partial k_y < 0)$ is satisfied for the equilibrium enstrophy spectrum for quasi-geostrophic turbulence and drift wave turbulence. The fundamental mechanism of zonal flow generation requires a synergy between local wave-wave interactions (in wavenumber space) and non-local couplings between waves and flows. Therefore, the zonal flow growth rate should depend on both the spectral structure of turbulence and properties of the zonal flow itself. From Eq. (10), we can see that the growth rate is indeed a function of wave spectrum $N_0(\boldsymbol{k})$ and zonal flow width q^{-1}. We will show in Sec. III that the growth rate also depends on ambient mean shear because mean shear decorrelates the couplings between wave and flows. The momentum transport associated with zonal flow modulational instability is given by

$$\frac{\partial}{\partial t}\delta V_x = \frac{\partial}{\partial y}\nu_t\frac{\partial}{\partial y}\delta V_x - \mu\delta V_x \simeq -q^2\nu_t\delta V_x - \mu\delta V_x, \qquad (11)$$

where the q dependence of turbulent viscosity ν_t is neglected. The leading behaviour of the zonal flow growth has the form of a negative viscosity instability

$$\nu_t = \int d^2k\frac{k_x^2k_y}{k^4}\frac{|\delta\omega_k|}{(\Omega_q - qv_{gy})^2 + \delta\omega_k^2}\frac{\partial N_0}{\partial k_y}. \qquad (12)$$

On the other hand, there are several processes by which the growth of zonal flow is limited or saturated. A non-perturbative mean field theory for the dynamics of minimum enstrophy relaxation shows that the PV flux consists of a positive hyper-viscosity term,[20] which reflects the saturation mechanism of zonal flows. The nonlinear interaction between zonal flows and a wave packet scatters the wave packet and returns energy back to turbulence. Moreover, as a seed zonal flow grows, it saturates the modulational instability by deflecting the propagation of wave packets. This mechanism is similar to the decorrelation of wave packets' trajectory by mean flow shearing, which we will discuss in Sec. III. From a energetics viewpoint, since energy is conserved between zonal flow and turbulence waves, the growth of zonal flow energy must saturate as wave energy is depleted.

III. EFFECTS OF SHEARED MEAN FLOW

The shearing field of tokamaks and the solar tachocline often include both zonal flow shears and mean flow shears. One of the interesting problems regarding the interplay among Rossby/drift waves, zonal flows, and a mean shear flow is the effect of a mean shear flow on the generation of zonal flows. This interplay among turbulence, zonal flows, and mean shears is shown schematically in Fig. 2. In the previous session, we have discussed how zonal flows grow by a long wave length modulation of wave turbulence. A mean flow shear can effectively decorrelate this modulation, thus inhibiting zonal flow growth. So as to address more realistic problems, in this section we study the modulational instability of test zonal flow in the presence of a sheared mean flow $V_0(y)\hat{x}$, focusing on calculating the suppression of zonal flow growth and PV transport.

In order to study the modulational instability in the ambient shearing field analytically, we use the method of characteristics to change coordinates to the shearing coordinates,[21] in which the wave kinetic equation becomes an ordinary differential equation. The ray trajectory of a wave packet in the presence of a mean zonal shear $V_0\hat{x}$ can be determined by eikonal theory,

$$\frac{dk_y}{dt} = -\frac{\partial\omega_k}{\partial y} = k_x\frac{\partial V_0}{\partial y},$$

$$\frac{dy}{dt} = \frac{\partial\omega_k}{\partial k_y} = v_{gy},$$

where the dispersion relation in the presence of mean shear is $\omega_k = k_xV_0 - \beta k_x/k^2$. For simplicity, a smooth mean shear $V_0(y) = -\Omega y$ is assumed, where Ω stands for the shearing
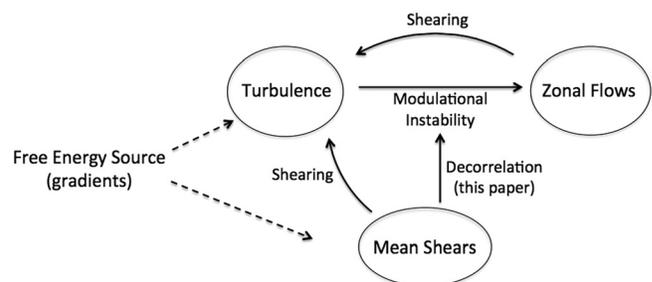
FIG. 2. Interplay among turbulence, zonal flows, and mean shears.

rate. Therefore, the meridional wave number k_y of wave packets increases linearly in time

$$k_y(t) = k_{y0} + k_x \Omega t. \tag{13}$$

The increase of k_y implies the increase of waves coupling to small scale dissipation. The increase of k_y also implies the decrease of meridional group velocity, which is given as $v_{gy} = 2\beta k_x k_y / k^4$. In other words, the growth of the effective wave inertia causes drift-Rossby wave packets to slow down, which in turn reduces the modulational response of waves. The meridional excursion of wave packets $e(t)$ is regulated by a mean shear as

$$y(t) = y(0) + e(t),$$

$$e(t) = \int_0^t \frac{2\beta k_x^2 \left(\frac{k_y(0)}{k_x} + \Omega t' \right)}{k_x^4 \left[\left(1 + \left(\frac{k_y(0)}{k_x} + \Omega^2 t' \right)^2 \right) \right]^2} dt'$$

$$= \frac{\beta}{k_x^2 \Omega} \left[\frac{1}{1 + \frac{k_y^2(0)}{k_x^2}} - \frac{1}{1 + \left(\frac{k_y(0)}{k_x} + \Omega t \right)^2} \right]. \tag{14}$$

From Eq. (14), we can see that the excursion is reduced by the mean shear. In this limit of $\Omega \to \infty$, wave packets are trapped ($e(t) \to 0$), with limited excursion. The linearized wave kinetic equation for a test zonal shear $\delta V_x' = \partial_y \delta V_x$ in a sheared mean flow is

$$\frac{\partial \tilde{N}}{\partial t} + v_{gy} \frac{\partial \tilde{N}}{\partial y} + k_x \Omega \frac{\partial \tilde{N}}{\partial k_y} + D \left(k_x^2 + k_y^2 \right) \tilde{N} = k_x \delta V_x' \frac{\partial N_0}{\partial k_y}, \tag{15}$$

where turbulent diffusion Dk^2 is used as an estimate of $\delta \omega_k$. $\delta \omega_k$ represents a relaxation of the wave action density N to the equilibrium N_0, in the absence of modulation. As we discussed in Sec. II, the wave action density and the potential enstrophy density are identical, up to a constant factor. We see that the wave action density represents the intensity field of PV and its evolution has a direct connection with PV mixing. Thus, $\delta \omega_k$ represents the decay due to forward enstrophy cascade and PV mixing. $\delta \omega_k$ is approximated as the characteristic time of PV mixing, which leads to the relaxation of N. Note that when we look at the phenomenon of zonal flow generation from the standpoint of PV mixing, it is the forward enstrophy cascade, not the inverse energy cascade, which is critical. The enstrophy forward cascade is analogous to induced diffusion to high wave number. The nonlinear coupling to small scale dissipation by the forward cascade effectively replaces the molecular viscosity by an eddy viscosity. The evolution of potential enstrophy by nonlinear interaction can be represented by a mixing process: $\langle \underline{v} \cdot \nabla q^2 \rangle \sim \tau_k^{-1} \langle \tilde{q}^2 \rangle_k$, where τ_k is the characteristic mixing time. We use eddy viscosity to model the mixing related to forward enstrophy cascade, and so we have $\tau_k^{-1} \simeq \delta \omega_k \simeq k^2 D_k$. We have included the explanation of this approximation to the revised manuscript.

We now use the method of characteristics to calculate the modulational response $\tilde{N}_k$. The first step is to perform a change of variables to shearing coordinates

$$\zeta = k_y - k_x \Omega t,$$
$$\xi = y - e(t).$$

Transforming the partial derivatives to the new coordinate: $\partial_t \to \partial_t + \zeta_t \partial_\zeta + \xi_t \partial_\xi$, $\partial_{k_y} \to \partial_\zeta$, $\partial_y \to \partial_\xi$, the Green's function of Eq. (15) in the shearing coordinate, $G(\xi, \zeta, k_x, t; \xi_1, \zeta_1, k_{1x}, t_1)$, is determined by

$$\partial_t G + D(k_x^2 + k_y^2)G = \delta(t - t_1)\delta(\xi - \xi_1)$$
$$\times \delta(k_x - k_{1x})\delta(\zeta - \zeta_1). \tag{16}$$

Defining α so that $\partial_t(\alpha t) = D(k_x^2 + k_y^2)$, the left hand side of Eq. (16) can be written as $e^{-\alpha t} \partial_t [e^{\alpha t} G]$. Solving for α,

$$\alpha t = \int_0^t dt' \, D \left[k_x^2 + \left(k_y(0) + k_x \Omega t' \right)^2 \right]$$

$$= D \left[k_x^2 t + k_y^2(0)t + k_y(0)k_x \Omega t^2 + \frac{k_x^2 \Omega^2 t^3}{3} \right]$$

$$= D \left[k_x^2 t + \frac{k_y(t)^3}{3\Omega k_x} - \frac{k_y^3(0)}{3\Omega k_x} \right],$$

the Green's function is derived as

$$G = \delta(\xi - \xi_1)\delta(\zeta - \zeta_1)\delta(k_x - k_{1x})$$

$$\times \exp \left\{ -D \left[\left(k_x^2 + k_{y0}^2 \right)t + k_x k_{y0} \Omega t^2 + \frac{k_x^2 \Omega^2}{3} t^3 \right] \right\}$$

$$\times \exp \left\{ D \left[\left(k_{1x}^2 + k_{y0}^2 \right)t_1 + k_{1x} k_{y0} \Omega t_1^2 + \frac{k_{1x}^2 \Omega^2}{3} t_1^3 \right] \right\}. \tag{17}$$

Changing variables back to the original frame, the Green's function becomes

$$g(y, k_y, k_x, t; y_1, k_{1y}, k_{1x}, t_1)$$
$$= \delta(y - y_1 - e(t) + e(t_1))\delta(k_y - k_{1y} - k_x \Omega(t - t_1))$$

$$\times \delta(k_x - k_{1x}) \times \exp \left[D \left(k_x^2 t + \frac{k_y^3}{3\Omega k_x} \right) \right]$$

$$\times \exp \left[D \left(k_{1x}^2 t_1 + \frac{k_{1y}^3}{3\Omega k_x} \right) \right]. \tag{18}$$

Thus, we can obtain the modulational enstrophy density induced by a test zonal shear in the presence of a mean shear as

$$\tilde{N}_k = \int\int\int dt_1 dy_1 dk_1^2 g(y, k_y, k_x, t; y_1, k_{1y}, k_x, t_1)$$

$$\times k_{1x} \delta V'_x(t_1, y_1) \frac{\partial N_0}{\partial k_y}$$

$$= \int_0^\infty d\tau e^{-Dk^2 Q \tau} \delta \hat{V}'_x e^{-i\Omega_q t_1 + iq(y - e(t) + e(t_1))} k_x \partial_{k_y} N_0, \tag{19}$$

where

$$\tau = t - t_1,$$

$$Q = 1 - \frac{k_x k_y}{k^2} \Omega \tau + \frac{k_x^2}{3k^2} \Omega^2 \tau^2.$$

Note that when there is no mean shear, we have $Q = 1$ and $y = v_{gy}t$. Therefore, Eq. (19) reduces to Eq. (9). Considering the long time limit ($\Omega t \gg 1$), the modulation of the enstrophy density $\tilde{N}_k$ becomes

$$\tilde{N}_k = \int_0^\infty d\tau e^{-\frac{\tau^3}{\tau_c^3} + i\Omega_q \tau + i\frac{q\beta}{k_0^2\Omega}} k_x \delta\hat{V}_x' \partial_{k_y} N_0, \qquad (20)$$

where the effective decorrelation rate of $\tilde{N}_k$ is defined as $\tau_c^{-1} \simeq \left(\frac{Dk_x^2\Omega^2}{3}\right)^{1/3}$. The decorrelation rate due to a random walk, (which is Dk^2 in the absence of shearing field), is enhanced by the coupling of shearing to turbulent decorrelation. In the limit of strong mean shear ($\Omega \gg Dk^2$), the corresponding $\tilde{N}_k$ becomes

$$\tilde{N}_k \simeq \left[\Gamma\left(\frac{4}{3}\right)\tau_c - \frac{\Omega_q^2}{6}\tau_c^3\right] e^{i\frac{q\beta}{k_0^2\Omega}} k_x \delta\hat{V}' \partial_{k_y}N_0$$
$$\simeq \left(\frac{3}{Dk_x^2\Omega^2}\right)^{1/3} e^{i\frac{q\beta}{k_0^2\Omega}} k_x \delta\hat{V}' \partial_{k_y}N_0 + O(\Omega^{-2}). \qquad (21)$$

The growth rate of zonal flow in strong mean shear limit is given as

$$\gamma_q \simeq -q^2 \int d^2k \frac{k_x^2 k_y}{k^4} \left(\frac{3}{Dk_x^2\Omega^2}\right)^{1/3} \frac{\partial N_0}{\partial k_y}. \qquad (22)$$

Compared with the system without ambient mean shear in Eq. (10), a strong mean shear reduces the growth of zonal flow by $(\Omega/Dk_x^2)^{-2/3}$. Since vorticity flux is equal to Reynolds force, which drives zonal flow, vorticity flux is reduced by the mean shear as

$$\langle \tilde{v}_y \nabla^2 \tilde{\psi} \rangle \simeq -q^2 \delta V \int d^2k \frac{k_x^2 k_y}{k^4} \left(\frac{3}{Dk_x^2\Omega^2}\right)^{1/3} \frac{\partial N_0}{\partial k_y}. \qquad (23)$$

As a consequence, the turbulent viscosity scales with the mean flow shearing rate to $\Omega^{-2/3}$ in strong shear limit

$$\nu_t(\Omega) \simeq \int d^2k \frac{k_x^2 k_y}{k^4} \left(\frac{3}{Dk_x^2\Omega^2}\right)^{1/3} \frac{\partial N_0}{\partial k_y}. \qquad (24)$$

Because the effective viscosity is negative, the decreasing magnitude of the viscosity corresponds to the suppression of zonal flow growth. In the theory of wave kinetics, wave packets mix PV and generating zonal flow. As the decorrelation rate of wave packets is enhanced by a mean shear, PV mixing by wave turbulence ultimately becomes less efficient. We can also see the inhibition of wave propagation by a strong mean shear from the excursion of the wave packets, which is suppressed, as shown in Eq. (14), and from the group velocity of Rossby wave $v_{gx} = 2\beta k_x k_y k^{-4}$, which is inversely proportional to shearing by the factor of Ω^{-3}.

It is worth proving that the rate of energy transfer from waves to zonal flows is reduced by a mean shear, while the sum of wave energy and zonal flow energy is conserved in

the presence of a mean shear. Since wave action density is conserved, wave energy density evolves according to

$$\frac{d}{dt}\varepsilon_k \simeq \frac{d\omega_k}{dt}\left(\frac{N_k}{-\beta k_x}\right) \simeq \frac{v_{gy}}{\beta}\langle \delta V_x' \tilde{N}_k \rangle, \qquad (25)$$

where $N_k/(-\beta k_y)$ is the wave action density (N_k is renormalized to enstrophy density in this paper). We use the $\tilde{N}_k$ derived in limit of strong mean shear to obtain

$$\frac{d}{dt}\varepsilon_k \simeq \int dq\, q^2 \delta\hat{V}_x^2 \frac{2k_x^2 k_y}{k^4}\left(\frac{3}{Dk_x^2\Omega^2}\right)^{1/3}\frac{\partial N_0}{\partial k_y}. \qquad (26)$$

On the other hand, the evolution of zonal flow energy in a strong mean shear is given by

$$\frac{\partial}{\partial t}\delta V_x^2 \simeq -2q^2\nu_t(\Omega)\delta V_x^2$$
$$\simeq -2q^2\delta V_x^2 \int d^2k \frac{k_x^2 k_y}{k^4}\left(\frac{3}{Dk_x^2\Omega^2}\right)^{1/3}\frac{\partial N_0}{\partial k_y}. \qquad (27)$$

Equations (26) and (27) show that the total energy of wave packets and zonal flows is conserved in the presence of a strong mean shear,

$$\frac{d}{dt}\left(\sum_k \varepsilon_k + \sum_q \delta V_{xq}^2\right) = 0. \qquad (28)$$

This is because the mean shear is assumed stable in this model ($d_t\Omega = 0$). However, the rate of energy transfer from turbulence to zonal flows is reduced by a mean shear with the scaling of $\Omega^{-2/3}$.

IV. CONCLUSION

We have investigated the important issue of how mean shear flows affect the transport of PV and zonal flow generation. For the systems of interest (magnetized plasma and quasi-geostrophic fluids), the conservation of PV is an essential characteristic of drift-Rossby wave dynamics, and the spatial mixing of PV is a fundamental mechanism for large scale flow generation. In a system with no mean shears, modulational stability calculations show that small scale wave packets are unstable to long wavelength perturbations, i.e., seed zonal flows. The seed zonal flow enforces turbulent PV transport, which reinforces the growth of the seed zonal flow. When a mean shear flow is introduced to the system, the growth rate of zonal flows is reduced because the correlation between wave packets and zonal flows is weakened. By inhibiting PV mixing or reducing the cross-phase of the Reynolds stress, mean shear has a significant influence on zonal flow dynamics. We demonstrate that in the strong shear limit, the zonal flow growth rate as well as PV flux decreases as $\Omega^{-2/3}$.

The results allow an improved interpretation of feedback of mean shear on modulational instability driving zonal flow in the L-H transition. Our model gives detailed dependence of the turbulent viscosity on mean shear and therefore gives a modification to the zonal flow evolution equation in the

L-H transition models (e.g., Refs. 6 and 7). Another application of the results would be to give a detailed prediction of the interplay between turbulence-driven shears and mean shears in I-phase. Here, I-phase refers to an intermediate, oscillatory phase between the L mode and the H mode, which occurs when the input power is near the H-mode power threshold. Understanding of the interaction between turbulence, zonal flows, and mean shears helps to elucidate the duration of I-phase. This is essential for the studies of nonlinear turbulent energy transfer (e.g., Refs. 22 and 23).

PV flux and turbulent viscosity are shown to be complex functions of turbulence spectrum (k, N_0), the structure of zonal flow (q), and nonlinear wave-wave self-decorrelation rate under modulation ($\delta\omega$). The main effect of a strong mean shear flow ($\Omega \gg \delta\omega$) on turbulent momentum transport in our model is to enhance modulational decorrelation rate from $\delta\omega$ to $(\delta\omega\Omega^2)^{1/3}$ (Table I). Based on the results, we suggest that momentum transport in the solar tachocline is non-Fickian, and the effect of mean solar differential rotation needs to be considered. There are other mechanisms which may affect turbulent momentum transport in the tachocline, including, but not limited to: reduction of turbulence intensity by mean and random shearing field, generation of turbulence by convective overshoot or shear, and reduction of turbulent transport by a magnetic field.

The effect of the magnetic field might possibly play a role in the tachocline. However, the plasma beta, defined as the ratio of the thermal pressure to the magnetic pressure, is expected to be very high ($\beta \gg 1$) in the tachocline, so the effect of magnetic field is uncertain, and most importantly, there is no direct observation of magnetic field in the tachocline. Therefore, the hydrodynamic model is meaningful. As we discussed in the Introduction, it is still unclear how the tachocline evolves under the meridional circulation-driven burrowing. Different mechanisms which oppose the burrowing during solar evolution have been proposed. There are two principal mechanisms: (1) turbulent viscous mixing of horizontal velocity, as in the Spiegel-Zahn[8] scenario, and (2) a fossil magnetic field in the radiative interior, as in the Gough-McIntyre[10] scenario. Spiegel and Zahn[8] suggest that stably stratified turbulence efficiently diffuses angular momentum in the latitudinal direction (i.e., positive turbulent viscosity) and so prevents the spreading of the tachocline. However, Gough and McIntyre[10] point out that the turbulence in a stably stratified rotating layer does not act as to diffuse the latitudinal gradient of angular velocity. Instead the turbulence would mix PV and so drive mean flows. They propose a hypothetical fossil magnetic field in the radiation zone to oppose the burrowing of the tachocline. Because the fossil magnetic field is localized in the radiative interior, it does not directly influence the turbulent transport in the tachocline; it

simply sets the boundary condition of the tachocline. In both the Spiegel-Zahn and Gough-McIntyre scenarios, turbulent transport in a stably stratified rotating layer plays a central role. This supports the need for a hydrodynamic analysis of turbulent mixing of PV in the tachocline.

One purpose of this work is to improve the hydrodynamic model of Spiegel-Zahn. The weak points of the Spiegel-Zahn model are the assumption of diffusion/mixing of horizontal momentum instead of PV, and the assumption of constant viscosity. Our model analyzes the inhomogeneous mixing of PV and includes the interaction between turbulence, turbulence-generated flows, and the background shear flows. We show that the turbulent viscosity is not constant; it depends on turbulence spectrum and flow structures. Moreover, the horizontal turbulent viscosity is found to be negative, i.e., tachocline burrowing is not balanced by horizontal turbulent momentum transport. Thus, the results suggest the need of other mechanisms to prevent tachocline penetration, and also point toward the (ultimately) critical role of jet frictional damping as dissipation. Our improved interpretation of the horizontal momentum transport could be used to develop a more accurate hydrodynamic tachocline model. While a 3D model for tachocline is beyond the scope of this work, we think it is worthwhile to couple the horizontal and the vertical fluxes of angular momentum related to Rossby wave turbulence. In other words, it would be interesting to revisit the Spiegel-Zahn model using the horizontal turbulent viscosity derived from this work. That, however, is a lengthy calculation which is beyond the scope of this paper.

In this work, we were concerned with the effects of mean shear on turbulence correlation times and thus turbulent fluxes and transport. However, mean shears can also reduce turbulent transport by altering the intensity of the turbulent fluctuations. Kim and Diamond[24] reconsider the effect of a strong mean shear flow on the transport of a passive scalar field. They show that the square amplitude of the turbulence scales with the mean shear as $\Omega^{-5/3}$ for a random flow with a localized frequency spectrum, while the flux of the passive scalar varies as Ω^{-1}. Their analytical results are confirmed by the calculations and numerical study by Leconte et al.[25] In the study of interchange and ion temperature gradient turbulence models,[26] it is shown that a strong reduction in transport of particles and heat results from a severe reduction in the amplitude of turbulent velocity. Those results indicate the importance of the turbulent transport suppression through the reduction in the turbulence intensity. Note that there is a fundamental difference between the mixing of passive scalars and active scalars. Back-reaction of mixed quantities on mixing carriers only occur for active scalars, like PV.

Like the majority of mean-field models, the scale separation in space and/or time is an underlying assumption of our perturbative analyses. The assumption simplifies the problem because it enables the use of the adiabatic invariant to calculate the modulational response of wave spectrum to a test zonal flow. The scale separation clearly exists between meso-scale, low frequency zonal flows and micro-scale higher frequency drift waves in magnetized plasma systems. For solar tachocline, there is no observation from which to determine whether such scale separation is valid or not. It is

TABLE I. Reduction of momentum transport by strong mean shear.

Turbulent viscosity	$\nu_t \simeq \int d^2k \, \dfrac{k_x^2 k_y}{k^4} R \dfrac{\partial N_0}{\partial k_y}$
Without mean shear	$R = \dfrac{|Dk^2|}{(\Omega_q - qv_{gy})^2 + (Dk^2)^2}$
With strong mean shear	$R = \left(\dfrac{3}{Dk_x^2\Omega^2} \right)^{1/3}$

nevertheless not an unreasonable assumption, considering that the scales of turbulence excited by penetrative convection are of the same order as solar granulation. For systems with no scale separation, it is more difficult to derive the momentum transport coefficients theoretically. One may approach the problem via the renormalization-group technique,[27] but applying this method to analyze turbulent momentum transport is still in the developing stage and is beyond the scope of this paper.

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